§6 Application to Residue

I) Evaluation of Improper Integrals :

- If $\lim_{R\to+\infty}\int_0^R f(x)dx$ exists, then we denote it by $\int_0^{+\infty} f(x)dx$.

- If both $\lim_{R_1\to+\infty}\int_0^{R_1} f(x)dx$ and $\lim_{R_2\to-\infty}\int_{R_2}^0 f(x)dx$ exist, then we denote it by

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{R_1\to+\infty}\int_0^{R_1} f(x)dx + \lim_{R_2\to-\infty}\int_{R_2}^0 f(x)dx$$

- $\text{P.V.}\int_{-\infty}^{+\infty} f(x)dx := \lim_{R\to+\infty}\int_{-R}^{R} f(x)dx$ (Cauchy principal value)

Why? Consider $f(x) = \begin{cases} 1 & \\ 0 & \text{if} \\ -1 & \end{cases} \begin{matrix} x>0 \\ x=0 \\ x<0 \end{matrix}$

$\lim_{R\to+\infty}\int_{-R}^{R} f(x)dx = 0$ but $\lim_{R\to+\infty}\int_{-R}^{R^2} f(x)dx = \lim_{R\to+\infty} R^2-R = +\infty$ (Does NOT exist)

(Speed of going to $+\infty / -\infty$ matters)

Suppose $f(x)$ is an even function, i.e. $f(-x)=f(x)\ \forall x\in\mathbb{R}$.

$2\int_0^R f(x)dx = \int_{-R}^R f(x)dx$ and so $2\int_0^{+\infty} f(x)dx$ exists

iff $\lim_{R\to+\infty}\int_{-R}^{R} f(x)dx = \text{P.V.}\int_{-\infty}^{+\infty} f(x)dx$ exists.

$\therefore \int_0^{+\infty} f(x)dx = \frac{1}{2}\text{P.V.}\int_{-\infty}^{+\infty} f(x)dx$

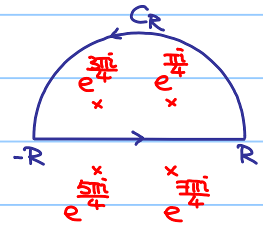
e.g. Find $\int_0^{+\infty} \frac{x^2}{x^4+1}dx$. Note $f(x) = \frac{x^2}{x^4+1}$ is even.

$\int_0^{+\infty} \frac{x^2}{x^4+1}dx$ exists iff P.V. $\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}dx$ exists.

Let $f(z) = \frac{z^2}{z^4+1}$

Residue theorem :

$$\underbrace{\int_{-R}^{R} f(z)\,dz}_{\overset{\parallel}{\int_{-R}^{R} f(x)dx}} + \int_{C_R} f(z)\,dz = 2\pi i \left(\operatorname*{Res}_{z=e^{\frac{\pi i}{4}}} f(z) + \operatorname*{Res}_{z=e^{\frac{3\pi i}{4}}} f(z) \right) \quad \text{— } (*)$$



poles of $f(z)$

(all of them are simple poles)

$(*)$ holds if $R>1$

- $f(z) = \frac{\overset{g(z)}{z^2}}{\underset{h(z)}{z^4+1}}$

$$\operatorname*{Res}_{z=e^{\frac{\pi i}{4}}} f(z) = \frac{g(e^{\frac{\pi i}{4}})}{h'(e^{\frac{\pi i}{4}})} = \frac{1}{4}e^{-\frac{\pi i}{4}}$$

Similarly, $\operatorname*{Res}_{z=e^{\frac{3\pi i}{4}}} f(z) = \frac{1}{4}e^{-\frac{3\pi i}{4}}$

- $\left|\int_{C_R} f(z)dz\right| \le \underset{\underset{0}{\downarrow}}{\pi R \cdot \frac{R^2}{R^4-1}}$ as $R \to +\infty$

On C_R , $|z^4+1| \ge |z^4| - 1 = R^4 - 1$

$\therefore \frac{1}{|z^4+1|} \le \frac{1}{R^4-1}$

$\left|\frac{z^2}{z^4+1}\right| \le \frac{R^2}{R^4-1}$

Let $R \to +\infty$ in $(*)$,

$$\lim_{R\to+\infty} \int_{-R}^{R} f(x)dx = 2\pi i\left(\frac{1}{4}e^{-\frac{\pi i}{4}} + \frac{1}{4}e^{-\frac{3\pi i}{4}}\right) = \frac{\pi}{\sqrt{2}}$$

- $\int_{-\infty}^{+\infty} f(x)\sin ax\,dx$, $\int_{-\infty}^{+\infty} f(x)\cos ax\,dx$

Trouble : Consider $f(z)\sin az$ on C_R ,

$$\sin az = \frac{e^{iaz}-e^{-iaz}}{2i} \Rightarrow \text{No good control of } |\sin az|$$

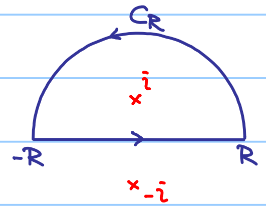
Trick : Consider $\int_{-R}^{R} f(x)e^{iax}dx = \int_{-R}^{R} f(x)\cos ax\,dx + i\int_{-R}^{R} f(x)\sin ax\,dx$

e.g. Find $\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x^2+1)^2}dx$.

Let $f(z) = \frac{1}{(z^2+1)^2}$

Residue theorem :

$$\int_{-R}^{R} f(x)e^{i3x}dx + \int_{C_R} f(z)e^{i3z}dz = 2\pi i\left(\operatorname*{Res}_{z=i} f(z)e^{i3z}\right) \text{ —— } (*)$$



poles of $f(z)e^{i3z}$

(all of them are poles of order 2)

- $f(z)e^{i3z} = \frac{e^{i3z}}{(z^2+1)^2} = \frac{e^{i3z}}{(z-i)^2(z+i)^2}$

$$= \frac{\phi(z)}{(z-i)^2}$$

where $\phi(z) = \frac{e^{i3z}}{(z+i)^2}$ is analytic at i (everywhere except $-i$)

Talyor series at i : $\phi(z) = \phi(i) + \phi'(i)(z-i) + \cdots$

- $f(z)e^{i3z} = \frac{\phi(z)}{(z-i)^2}$

$$= \frac{\phi(i)}{(z-i)^2} + \boxed{\frac{\phi'(i)}{z-i}} + \cdots$$

(What we care !

$\therefore \operatorname*{Res}_{z=i} f(z)e^{i3z} = \phi'(i) = \frac{1}{ie^3}$

- $\left|\int_{C_R} f(z)e^{i3z}dz\right| \le \pi R \cdot 1 \cdot \frac{1}{(R^2-1)^2}$

$\downarrow$

0 as $R \to +\infty$

On C_R , $|z^2+1| \ge |z^2| - 1 = R^2 - 1$

$\therefore \frac{1}{|z^2+1|} \le \frac{1}{R^2-1}$

$\left|\frac{1}{(z^2+1)^2}\right| \le \frac{1}{(R^2-1)^2}$

$|e^{i3z}| \le |e^{-3\operatorname{Im}(z)}| \le 1$

Let $R \to +\infty$ in $(*)$,

$$\lim_{R\to+\infty} \int_{-R}^{R} f(x)e^{-i3x}dx = 2\pi i\left(\frac{1}{ie^3}\right) = \frac{2\pi}{e^3}$$

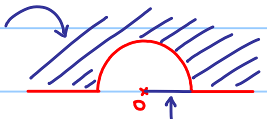
$\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x^2+1)^2}dx = \frac{2\pi}{e^3}$ and $\int_{-\infty}^{+\infty} \frac{\sin 3x}{(x^2+1)^2}dx = 0$

↑ Odd function ↑ $\therefore$ NOT surprising

Generalization of the Estimation

Jordan's Lemma:

f(z) is analytic



sufficiently large.

$C_R = \{ z = Re^{i\theta} : 0 \le \theta \le \pi \}$

Suppose $|f(z)| \le M_R$ $\forall z \in C_R$ and $\lim\limits_{R\to+\infty} M_R = 0$.

(Note: depends on R)

Then $\lim\limits_{R\to+\infty} \int_{C_R} f(z) e^{iaz} dz = 0$ where $a>0$.

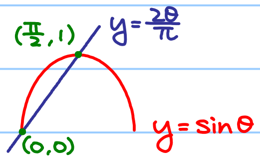
But, some trouble:

ML-estimate $\Rightarrow$ $\left|\int_{C_R} f(z) e^{iaz} dz\right| \underset{(*)}{\le} \overset{+\infty}{\overset{\uparrow}{\pi R}} \cdot \underset{\underset{0}{\downarrow}}{M_R} \cdot 1$ $\qquad |e^{iaz}| \le |e^{-a\mathrm{Im}(z)}| \le 1$

when $R\to+\infty$

That means we have to find a better estimation instead of (*)

Jordan's inequality: $\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{R}$



for $0 \le \theta \le \frac{\pi}{2}$

$$\sin\theta \ge \frac{2\theta}{\pi}$$

$$e^{-R\sin\theta} \le e^{-\frac{2R\theta}{\pi}}$$

$$\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta \le \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}} d\theta = \frac{\pi}{2R}(1-e^{-R}) < \frac{\pi}{2R}$$

Also $\int_0^{\pi} e^{-R\sin\theta} d\theta = \int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta + \underbrace{\int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta} d\theta}_{\text{let } \alpha=\pi-\theta}$

$= 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} d\theta$

$< 2\cdot\frac{\pi}{2R} = \frac{\pi}{R}$

proof of Jordan's lemma:

$$\left|\int_{C_R} f(z)e^{iaz}dz\right|$$

$$= \left|\int_0^{\pi} f(Re^{i\theta})\; e^{ia(Re^{i\theta})} \; iRe^{i\theta}d\theta\right| \qquad z = Re^{i\theta}$$

$$\le \int_0^{\pi} |f(Re^{i\theta})|\cdot|e^{ia(Re^{i\theta})}|\cdot|iRe^{i\theta}|\,d\theta \qquad dz = iRe^{i\theta}d\theta$$

$$\le M_R\cdot R\cdot \int_0^{\pi} e^{-aR\sin\theta}d\theta$$

$$< M_R\cdot R\cdot \frac{\pi}{\underbrace{aR}_{\text{Jordan's lemma provides } \frac{1}{R}.}}$$

$$= \underset{\substack{\downarrow\\ 0}}{M_R}\cdot\frac{\pi}{a}$$

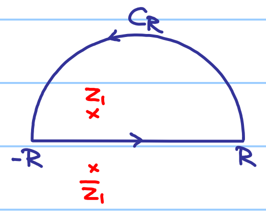
as $R\to+\infty$

e.g. Find $\text{P.V.}\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+2x+2}dx$, $\text{P.V.}\int_{-\infty}^{+\infty}\frac{x\cos x}{x^2+2x+2}dx$

Let $f(z) = \frac{z}{z^2+2z+2} = \frac{z}{(z-z_1)(z-\bar{z}_1)}$ where $z_1 = -1+i$

Residue theorem:

$$\int_{-R}^{R} f(x)e^{ix}dx + \int_{C_R} f(z)e^{iz}dz = 2\pi i\left(\operatorname*{Res}_{z=z_1} f(z)e^{iz}\right) \quad \text{---} (*)$$



poles of $f(z)e^{iz}$

(all of them are simple poles)

- Ex: $\operatorname*{Res}_{z=z_1}(f(z)e^{iz}) = \frac{z_1e^{iz_1}}{z_1-\bar{z}_1}$

- On C_R, $|f(z)| = \frac{|z|}{|z-z_1||z-\bar{z}_1|} \le \frac{R}{(R-\sqrt{2})^2} \longrightarrow 0$ as $R\to+\infty$

ML-estimate $\Rightarrow \left|\int_{C_R} f(z)e^{iz}dz\right| \le \pi \overset{\substack{+\infty\\ \uparrow}}{R}\cdot \underset{\substack{\downarrow\\ 0}}{M_R}$ (useless)

as $R\to+\infty$

But Jordan's lemma $\Rightarrow \int_{C_R} f(z)e^{iz}dz \to 0$ as $R\to+\infty$.

- $\int_0^{\overset{(\pi)}{2\pi}} F(\sin\theta, \cos\theta)\, d\theta$

Let $z = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$

$(z = e^{i2\theta} \quad 0 \le \theta \le \pi)$

Then $\sin\theta = \frac{z - z^{-1}}{2i}$, $\cos\theta = \frac{z + z^{-1}}{2}$, $d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

$$= \int_C \underbrace{F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{1}{iz}}_{f(z)} d\theta$$ where C is the positively oriented unit circle

$$= 2\pi i \left(\sum_k \operatorname*{Res}_{z = z_k} f(z) \right)$$ sum over all poles inside C.

e.g. $\int_0^{2\pi} \frac{1}{5 + 4\sin\theta} d\theta$

$$= \int_C \frac{1}{2z + 5iz - 2} dz$$

$$= \int_C \frac{1}{(2z + i)(z + 2i)} dz$$

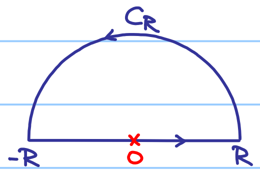
$$= 2\pi i \operatorname*{Res}_{z = -\frac{i}{2}} \frac{1}{(2z + i)(z + 2i)}$$

$$= \frac{2\pi}{3}$$

e.g. Find $\int_0^{+\infty} \frac{\sin x}{x} dx$.

Note : $\frac{\sin x}{x}$ is even , so we want to find P.V. $\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx$.

Then we consider $f(z) e^{iz}$ where $f(z) = \frac{1}{z}$ and the integral along the following contour



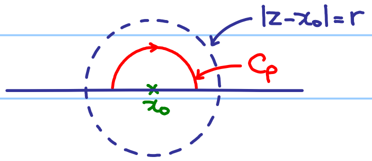
Trouble : 0 is a simple pole of $f(z) e^{iz} = \frac{e^{iz}}{z}$ which lies on the contour !

The following lemma can help to deal with that !

Lemma : Suppose $f(z)$ has a simple poles at $z=x_0\in\mathbb{R}$ and $f(z)$ has a Laurent series representation in a punctured disk $0<|z-x_0|<r$

i.e. $f(z)=\frac{C_{-1}}{z-x_0}+C_0+C_1(z-x_0)+\cdots$

Then, $\lim_{\rho\to 0}\int_{\rho} f(z)dz = -C_{-1}\pi i$



proof : We can express $f(z)=\frac{C_{-1}}{z-x_0}+g(z)$ where $g(z)$ is analytic at x_0.

$\Downarrow$

$g(z)$ is bounded in $\{|z-x_0|<\rho_0\}$ for some $\rho_0>0$

i.e. $|g(z)|\le M$ for some $M>0$.

$$\int_{C_\rho} f(z)dz = \int_{C_\rho}\frac{C_{-1}}{z-x_0}+g(z)\,dz$$

- $\left|\int_{C_\rho} g(z)dz\right| \le \pi\rho\cdot M \to 0$ as $\rho\to 0$
- $\int_{C_\rho}\frac{C_{-1}}{z-x_0}dz$ Let $z=x_0+\rho e^{i\theta}$, $0\le\theta\le\pi$

$= -\int_0^{\pi}\frac{C_{-1}}{\rho e^{i\theta}}\,i\rho e^{i\theta}d\theta$ $\quad dz=i\rho e^{i\theta}d\theta$

$\because C_\rho$ goes in clockwise direction.

$= -\int_0^{\pi} C_{-1}\,i\,d\theta$

$= -C_{-1}\pi i$

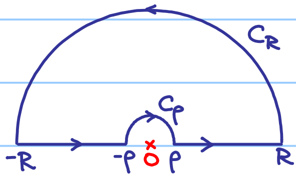
$\therefore \lim_{\rho\to 0}\int_{\rho} f(z)dz = -C_{-1}\pi i$

Come back to the example :

Find $\int_0^{+\infty}\frac{\sin x}{x}dx$.

By residue theorem,

$$\int_{-R}^{-\rho}\frac{e^{ix}}{x}dx+\int_{C_\rho}\frac{e^{iz}}{z}dz+\int_{\rho}^{R}\frac{e^{ix}}{x}dx+\int_{C_R}\frac{e^{iz}}{z}dz=0$$



avoid the pole of $\frac{e^{iz}}{z}$.

- $\lim_{\rho\to 0}\int_{C_\rho}\frac{e^{iz}}{z}dz = -\pi i \operatorname{Res}_{z=0}\frac{e^{iz}}{z} = -\pi i$
- On C_R, $\left|\frac{1}{z}\right| = \frac{1}{R} \to 0$ as $R\to+\infty$

$\therefore \int_{C_R}\frac{e^{iz}}{z}dz \longrightarrow 0$ as $R\to+\infty$

$$\lim_{\rho\to 0}\lim_{R\to+\infty}\int_{-R}^{-\rho}\frac{e^{ix}}{x}dx+\int_{\rho}^{R}\frac{e^{ix}}{x}dx=\pi i$$

$$2\int_0^{+\infty}\frac{\sin x}{x}dx=\pi$$

$$\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$$

Integration along a branch cut

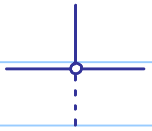
e.g. Find $\int_0^{+\infty} \frac{x^{-a}}{x+1}\,dx \qquad 0<a<1$

$\Downarrow$

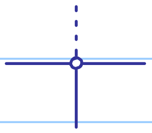
Let $f(z) = \frac{z^{-a}}{z+1}$

Trouble : z^{-a} is NOT well-defined on the whole complex plane.

Recall: Define $f_1(z) = e^{-a(\ln r + i\theta)} \qquad r>0 ,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$



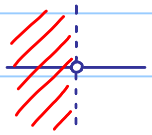
$f_2(z) = e^{-a(\ln r + i\theta)} \qquad r>0 ,\ \frac{\pi}{2} < \theta < \frac{5\pi}{2}$



$1-i = \sqrt{2}e^{i(-\frac{\pi}{4})} = \sqrt{2}e^{i(\frac{7\pi}{4})}$

$f_1(1-i) = e^{a(\ln\sqrt{2} - \frac{\pi}{4}i)}$ but $f_2(1-i) = e^{a(\ln\sqrt{2} + \frac{7\pi}{4}i)}$

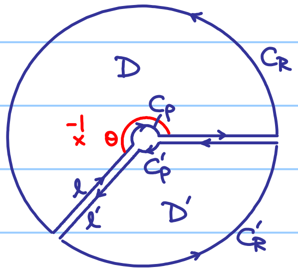
However, $f_1(z)$ and $f_2(z)$ agree on $\{z\in\mathbb{C} : \operatorname{Re}(z)<0\}$



Residue theorem

$$\Rightarrow \qquad \int_{D'} \frac{f_2(z)}{z+1}\,dz = 0$$

$$+) \qquad \int_{D} \frac{f_1(z)}{z+1}\,dz = 2\pi i \operatorname*{Res}_{z=-1}\left(\frac{f_1(z)}{z+1}\right) = f_1(-1) = e^{-a(\ln 1 + \pi i)} = e^{-a\pi i}$$



$$\left(\int_{C_R}\frac{f_1(z)}{z+1}dz + \int_{C_R'}\frac{f_2(z)}{z+1}dz\right) + \left(\int_{C_\rho}\frac{f_1(z)}{z+1}dz + \int_{C_\rho'}\frac{f_2(z)}{z+1}dz\right)$$

$$+\underbrace{\left(\int_{\ell}\frac{f_1(z)}{z+1}dz + \int_{\ell'}\frac{f_2(z)}{z+1}dz\right)}_{\overset{\shortparallel}{0}} + \left(\int_\rho^R\frac{f_1(z)}{z+1}dz + \int_R^\rho\frac{f_2(z)}{z+1}dz\right) = e^{-a\pi i}$$

since $f_1(z)$ and $f_2(z)$ agree

but ℓ and ℓ' are in opposite direction.

On C_R (C_R') $\quad \left|\frac{\overset{f_2(z)}{f_1(z)}}{z+1}\right| \le \frac{R^{-a}}{R-1} \quad \Rightarrow \left|\int_{\underset{C_R'}{C_R}} \frac{\overset{f_2(z)}{f_1(z)}}{z+1}dz\right| \le \underset{(2\pi-\theta)R}{\theta R}\cdot\frac{R^{-a}}{R-1} \longrightarrow 0$ as $R\to+\infty$

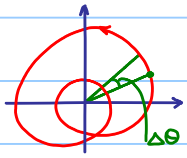
On C_ρ (C_ρ') $\quad \left|\frac{\overset{f_2(z)}{f_1(z)}}{z+1}\right| \le \frac{\rho^{-a}}{1-\rho} \quad \Rightarrow \left|\int_{\underset{C_\rho'}{C_\rho}} \frac{\overset{f_2(z)}{f_1(z)}}{z+1}dz\right| \le \underset{(2\pi-\theta)\rho}{\theta\rho}\cdot\frac{\rho^{-a}}{1-\rho} \longrightarrow 0$ as $\rho\to 0$

$$\int_\rho^R\frac{f_1(z)}{z+1}dz + \int_R^\rho\frac{f_2(z)}{z+1}dz = \int_\rho^R \frac{x^{-a}}{x+1} + \frac{x^{-a}e^{-2\pi i a}}{x+1}dx = (1-e^{-2\pi i a})\int_\rho^R\frac{x^{-a}}{x+1}dx$$

$\therefore$ By letting $R\to+\infty$, $\rho\to 0$, $\int_0^{+\infty}\frac{x^{-a}}{x+1}dx = \frac{2\pi i e^{-\pi i a}}{1-e^{-2\pi i a}} = \frac{\pi}{\sin\pi a}$

II) Winding Number and Rouché's Theorem

Suppose C is a closed contour in $\mathbb{C}$ and $0 \notin C$.



Idea : "$\Sigma \Delta\theta$" = change of angle

$$\int_C d\theta = 2k\pi \quad k \in \mathbb{Z}$$

where k is the number of turns that C makes around the origin.

Consider $z = re^{i\theta}$ $(\because z \neq 0)$

$$dz = e^{i\theta}dr + rie^{i\theta}d\theta$$

$$= \frac{1}{r}(re^{i\theta})dr + i(re^{i\theta})d\theta$$

$$\therefore \frac{dz}{z} = \frac{dr}{r} + i d\theta$$

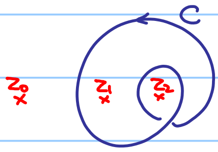
$$\int_C \frac{dz}{z} = \int_C \frac{dr}{r} + i\int_C d\theta \qquad \int_C \frac{dr}{r} = \int_C d\ln r = 0 \quad \because C \text{ is closed.}$$

$$= i\int_C d\theta \quad (= 2k\pi i)$$

Definition : Winding number of C around 0 is defined to be $\frac{1}{2\pi i}\int_C \frac{dz}{z}$

$(z=a)$ $\frac{1}{2\pi i}\int_C \frac{dz}{z-a}$ $a \notin C$

e.g.



Winding number of C around $z_0 = 0$

$z_1 = 1$

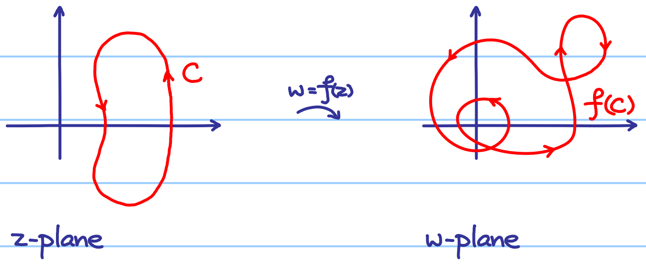
$z_2 = 2$.

Definition : A function is meromorphic in a domain D if it is analytic throughout D except a set of isolated poles.

C : simple closed contour

D : simply connected domain , $\partial D = C$

f : meromorphic in D, analytic and nonzero on C.



Note : $0 \notin f(C)$

Winding number of $f(C)$ around $w=0$

$= \int_{f(C)} \frac{dw}{w}$

$= \int_C \frac{1}{f(z)} \underbrace{\frac{dw}{dz}}_{f'(z)} dz$ (Change of variable)

$= \int_C \frac{f'(z)}{f(z)} dz$

By assumption + Residue theorem

$= \frac{1}{2\pi i} \cdot 2\pi i \sum_j \operatorname{Res}_{z=z_j} \left(\frac{f'(z)}{f(z)} \right)$ sum over poles of $\frac{f'(z)}{f(z)}$ inside C.

$= \sum_j \operatorname{Res}_{z=z_j} \left(\frac{f'(z)}{f(z)} \right)$

$= \sum_j m_j$

(sum of zeros and poles, count with multiplicities)

$= Z_f - P_f$

where $Z_f = \sum_{j;\, m_j>0} m_j$

$P_f = -\sum_{j;\, m_j<0} m_j$

Note: poles of $\frac{f'(z)}{f(z)}$: zeros and poles of f.

If z_j is a zero or a pole,

then we write $f(z) = (z-z_j)^{m_j} g(z)$

where g is analytic and nonzero at z_j

($m_j > 0$ if z_j is a zero ; $m_j < 0$ if z_j is a pole)

$f'(z) = m_j (z-z_j)^{m_j - 1} g(z) + (z-z_j)^{m_j} g'(z)$

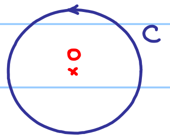
$\therefore \frac{f'(z)}{f(z)} = \frac{m_j}{z-z_j} + \frac{g'(z)}{g(z)}$ ($\frac{g'(z)}{g(z)}$ is analytic at z_j)

$\operatorname{Res}_{z=z_j} \frac{f'(z)}{f(z)} = m_j$

e.g. $f(z)=z^2$

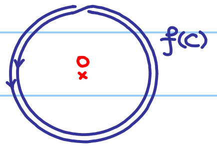
$C=\{z=e^{i\theta} : 0\le\theta\le 2\pi\}$

$= \text{unit circle}$

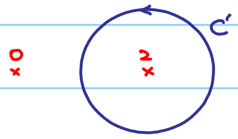


$f(C)=\{z=e^{2i\theta} : 0\le\theta\le 2\pi\}$

$= \text{unit circle (but winding number} = 2)$

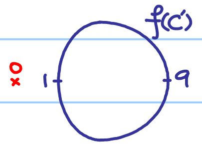


$C'=\{z=2+e^{i\theta} : 0\le\theta\le 2\pi\}$



$f(C')=\{(4+4\cos\theta+\cos 2\theta)+i(4\sin\theta+\sin 2\theta) : 0\le\theta\le 2\pi\}$

winding number = 0



Corollary: If f is analytic and nonzero on $D\cup\partial D$ (No zero and pole)

then winding number of $f(C)$ around $0 = 0$

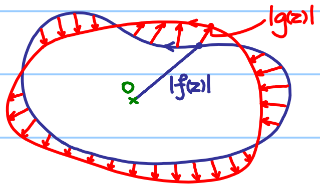
Rouché's Theorem: Let f, g be analytic functions inside an on a simple closed contour C,

and suppose that $|f(z)|>|g(z)|$ at each point on C.

Then $Z_f = Z_{f+g}$, in other words,

winding number of $f(C)$ around 0 = winding number of $(f+g)(C)$ around 0

proof:



$Z_{f+g}-Z_f$

= winding number of $f(C)$ around 0 − winding number of $(f+g)(C)$ around 0

$$= \frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}-\frac{f'}{f}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{fg'-gf'}{f(f+g)}\,dz \qquad \frac{g'f-gf'}{f^2}=\left(\frac{g}{f}\right)'=\left(1+\frac{g}{f}\right)'$$

$$= \frac{1}{2\pi i}\int_C \frac{\left(\frac{fg'-gf'}{f^2}\right)}{\left(1+\frac{g}{f}\right)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{\left(1+\frac{g}{f}\right)'}{\left(1+\frac{g}{f}\right)}\,dz$$

= winding number of $(1+\frac{g}{f})(C)$ around 0 $\qquad \left|\left(1+\frac{g}{f}\right)(z)-1\right| = \left|\frac{g(z)}{f(z)}\right| < 1$ for all $z\in C$.

$= 0$ $\qquad \therefore (1+\frac{g}{f})(C)$ lies inside $\{|w-1|<1\}$

winding number of $(1+\frac{g}{f})(C)$ around $0 = 0$

e.g. Determine number of roots of the equation $z^7 - 4z^3 + z - 1 = 0$ inside the unit circle C.

Let $f(z) = -4z^3$, $g(z) = z^7 + z - 1$

On C, $|f(z)| = 4$

$$|g(z)| \le |z^7| + |z| + 1 = 3 < |f(z)|$$

$$\therefore \underset{\overset{\shortparallel}{3}}{Z_f} = Z_{f+g}$$

e.g. (Revisit of Fundamental Theorem of Algebra)

Let $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$

$f(z) = z^n$, $g(z) = a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$

Consider $R > |a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| + 1$

Then, on $\{|z| = R\}$,

$$|f(z)| = R^n$$

$$\begin{aligned} |g(z)| &\le |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \cdots + |a_1|R + |a_0| \\ &\le |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-1} + \cdots + |a_1|R^{n-1} + |a_0|R^{n-1} \qquad (\because R > 1) \\ &= (|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|)R^{n-1} \\ &< R^n \end{aligned}$$

$$\therefore \underset{\overset{\shortparallel}{n}}{Z_f} = Z_{f+g} \quad \text{inside } \{|z| = R\}$$